8.3 Trigonometric Integrals

- **Solve trigonometric integrals involving powers of sine and cosine.**
- **Solve trigonometric integrals involving powers of secant and tangent.**
- **Solve trigonometric integrals involving sine-cosine products with different angles.**

SHEILA SCOTT MACINTYRE (1910–1960)

Sheila Scott Macintyre published her first paper on the asymptotic periods of integral functions in 1935. She completed her doctorate work at Aberdeen University, where she taught. In 1958 she accepted a visiting research fellowship at the University of Cincinnati.

Integrals Involving Powers of Sine and Cosine

In this section, you will study techniques for evaluating integrals of the form

$$\int \sin^m x \cos^n x \, dx \quad \text{and} \quad \int \sec^m x \tan^n x \, dx$$

where either m or n is a positive integer. To find antiderivatives for these forms, try to break them into combinations of trigonometric integrals to which you can apply the Power Rule.

For instance, you can evaluate

$$\int \sin^5 x \cos x \, dx$$

with the Power Rule by letting $u = \sin x$. Then, $du = \cos x \, dx$ and you have

$$\int \sin^5 x \cos x \, dx = \int u^5 \, du = \frac{u^6}{6} + C = \frac{\sin^6 x}{6} + C.$$

To break up $\int \sin^m x \cos^n x \, dx$ into forms to which you can apply the Power Rule, use the following identities.

$$\sin^2 x + \cos^2 x = 1 \qquad \text{Pythagorean identity}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \text{Half-angle identity for } \sin^2 x$$

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \text{Half-angle identity for } \cos^2 x$$

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING POWERS OF SINE AND COSINE

1. When the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then, expand and integrate.

$$\int \overbrace{\sin^{2k+1}}^{\text{Odd}} x \cos^n x \, dx = \int \overbrace{(\sin^2 x)^k}^{\text{Convert to cosines}} \cos^n x \overbrace{\sin x \, dx}^{\text{Save for } du} = \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

2. When the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then, expand and integrate.

$$\int \sin^m x \overbrace{\cos^{2k+1}}^{\text{Odd}} x \, dx = \int \sin^m x \overbrace{(\cos^2 x)^k}^{\text{Convert to sines}} \overbrace{\cos x \, dx}^{\text{Save for } du} = \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx\frac{1}{2}$$

3. When the powers of both the sine and cosine are even and nonnegative, make repeated use of the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to convert the integrand to odd powers of the cosine. Then proceed as in the second guideline.

EXAMPLE 1 Power of Sine Is Odd and Positive

Find $\int \sin^3 x \cos^4 x \, dx$.

Solution Because you expect to use the Power Rule with $u = \cos x$, *save one sine factor* to form du and convert the remaining sine factors to cosines.

$$\int \sin^3 x \cos^4 x \, dx = \int \sin^2 x \cos^4 x (\sin x) \, dx \qquad \text{Rewrite.}$$

$$= \int (1 - \cos^2 x) \cos^4 x \sin x \, dx \qquad \text{Trigonometric identity}$$

$$= \int (\cos^4 x - \cos^6 x) \sin x \, dx \qquad \text{Multiply.}$$

$$= \int \cos^4 x \sin x \, dx - \int \cos^6 x \sin x \, dx \qquad \text{Rewrite.}$$

$$= -\int \cos^4 x(-\sin x) \, dx + \int \cos^6 x(-\sin x) \, dx$$

$$= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C \qquad \text{Integrate.}$$

TECHNOLOGY A computer algebra system used to find the integral in Example 1 yielded the following.

$$\int \sin^3 x \cos^4 x \, dx = -\cos^5 x\left(\frac{1}{7}\sin^2 x + \frac{2}{35}\right) + C$$

Is this equivalent to the result obtained in Example 1?

In Example 1, *both* of the powers m and n happened to be positive integers. This strategy will work as long as either m or n is odd and positive. For instance, in the next example, the power of the cosine is 3, but the power of the sine is $-\frac{1}{2}$.

EXAMPLE 2 Power of Cosine Is Odd and Positive

See LarsonCalculus.com for an interactive version of this type of example.

Evaluate $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} \, dx$.

Solution Because you expect to use the Power Rule with $u = \sin x$, *save one cosine factor* to form du and convert the remaining cosine factors to sines.

$$\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} \, dx = \int_{\pi/6}^{\pi/3} \frac{\cos^2 x \cos x}{\sqrt{\sin x}} \, dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{(1 - \sin^2 x)(\cos x)}{\sqrt{\sin x}} \, dx$$

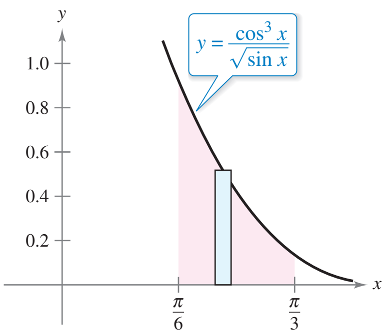
$$= \int_{\pi/6}^{\pi/3} \left[(\sin x)^{-1/2} - (\sin x)^{3/2}\right] \cos x \, dx$$

$$= \left[\frac{(\sin x)^{1/2}}{1/2} - \frac{(\sin x)^{5/2}}{5/2}\right]_{\pi/6}^{\pi/3}$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)^{1/2} - \frac{2}{5}\left(\frac{\sqrt{3}}{2}\right)^{5/2} - \sqrt{2} + \frac{\sqrt{32}}{80}$$

$$\approx 0.239$$

Figure 8.4 shows the region whose area is represented by this integral.



The area of the region is approximately 0.239.

Figure 8.4

EXAMPLE 3 Power of Cosine Is Even and Nonnegative

Find $\int \cos^4 x \, dx$.

Solution Because m and n are both even and nonnegative $(m = 0)$, you can replace $\cos^4 x$ by

$$\left(\frac{1 + \cos 2x}{2}\right)^2.$$

So, you can integrate as shown.

$$\begin{aligned}
\int \cos^4 x \, dx &= \int \left(\frac{1 + \cos 2x}{2}\right)^2 dx && \text{Half-angle identity} \\
&= \int \left(\frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}\right) dx && \text{Expand.} \\
&= \int \left[\frac{1}{4} + \frac{\cos 2x}{2} + \frac{1}{4}\left(\frac{1 + \cos 4x}{2}\right)\right] dx && \text{Half-angle identity} \\
&= \frac{3}{8}\int dx + \frac{1}{4}\int 2 \cos 2x \, dx + \frac{1}{32}\int 4 \cos 4x \, dx && \text{Rewrite.} \\
&= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C && \text{Integrate.}
\end{aligned}$$

Use a symbolic differentiation utility to verify this. Can you simplify the derivative to obtain the original integrand?

In Example 3, when you evaluate the definite integral from 0 to $\pi/2$, you obtain

$$\begin{aligned}
\int_0^{\pi/2} \cos^4 x \, dx &= \left[\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right]_0^{\pi/2} \\
&= \left(\frac{3\pi}{16} + 0 + 0\right) - (0 + 0 + 0) \\
&= \frac{3\pi}{16}.
\end{aligned}$$

Note that the only term that contributes to the solution is

$$\frac{3x}{8}.$$

This observation is generalized in the following formulas developed by John Wallis (1616–1703).

JOHN WALLIS (1616–1703)

Wallis did much of his work in calculus prior to Newton and Leibniz, and he influenced the thinking of both of these men. Wallis is also credited with introducing the present symbol (∞) for infinity.

See LarsonCalculus.com to read more of this biography.

Wallis's Formulas

1. If n is odd $(n \geq 3)$, then

$$\int_0^{\pi/2} \cos^n x \, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) \cdots \left(\frac{n-1}{n}\right).$$

2. If n is even $(n \geq 2)$, then

$$\int_0^{\pi/2} \cos^n x \, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \cdots \left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right).$$

These formulas are also valid when $\cos^n x$ is replaced by $\sin^n x$. (You are asked to prove both formulas in Exercise 88.)

Bettmann/Corbis

Integrals Involving Powers of Secant and Tangent

The guidelines below can help you evaluate integrals of the form

$$\int \sec^m x \tan^n x \, dx.$$

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING POWERS OF SECANT AND TANGENT

1. When the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then, expand and integrate.

$$\int \sec^{\overset{\text{Even}}{\widehat{2k}}} x \tan^n x \, dx = \int \overbrace{(\sec^2 x)^{k-1}}^{\text{Convert to tangents}} \tan^n x \overbrace{\sec^2 x \, dx}^{\text{Save for } du} = \int (1 + \tan^2 x)^{k-1} \tan^n x \sec^2 x \, dx$$

2. When the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then, expand and integrate.

$$\int \sec^m x \tan^{\overbrace{2k+1}^{\text{Odd}}} x \, dx = \int \sec^{m-1} x \overbrace{(\tan^2 x)^k}^{\text{Convert to secants}} \overbrace{\sec x \tan x \, dx}^{\text{Save for } du} = \int \sec^{m-1} x (\sec^2 x - 1)^k \sec x \tan x \, dx$$

3. When there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\int \tan^n x \, dx = \int \tan^{n-2} x \overbrace{(\tan^2 x)}^{\text{Convert to secants}} \, dx = \int \tan^{n-2} x (\sec^2 x - 1) \, dx$$

4. When the integral is of the form

$$\int \sec^m x \, dx$$

where m is odd and positive, use integration by parts, as illustrated in Example 5 in Section 8.2.

5. When none of the first four guidelines applies, try converting to sines and cosines.

EXAMPLE 4 **Power of Tangent Is Odd and Positive**

Find $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}} \, dx.$

Solution Because you expect to use the Power Rule with $u = \sec x$, *save a factor of $(\sec x \tan x)$* to form du and convert the remaining tangent factors to secants.

$$\begin{aligned}
\int \frac{\tan^3 x}{\sqrt{\sec x}} \, dx &= \int (\sec x)^{-1/2} \tan^3 x \, dx \\
&= \int (\sec x)^{-3/2} (\tan^2 x)(\sec x \tan x) \, dx \\
&= \int (\sec x)^{-3/2} (\sec^2 x - 1)(\sec x \tan x) \, dx \\
&= \int [(\sec x)^{1/2} - (\sec x)^{-3/2}](\sec x \tan x) \, dx \\
&= \frac{2}{3}(\sec x)^{3/2} + 2(\sec x)^{-1/2} + C
\end{aligned}$$

EXAMPLE 5 Power of Secant Is Even and Positive

Find $\int \sec^4 3x \tan^3 3x \, dx$.

Solution Let $u = \tan 3x$, then $du = 3 \sec^2 3x \, dx$ and you can write

$$\begin{aligned}
\int \sec^4 3x \tan^3 3x \, dx &= \int \sec^2 3x \tan^3 3x(\sec^2 3x) \, dx \\
&= \int (1 + \tan^2 3x) \tan^3 3x(\sec^2 3x) \, dx \\
&= \frac{1}{3}\int (\tan^3 3x + \tan^5 3x)(3 \sec^2 3x) \, dx \\
&= \frac{1}{3}\left(\frac{\tan^4 3x}{4} + \frac{\tan^6 3x}{6}\right) + C \\
&= \frac{\tan^4 3x}{12} + \frac{\tan^6 3x}{18} + C.
\end{aligned}$$

In Example 5, the power of the tangent is odd and positive. So, you could also find the integral using the procedure described in the second guideline on page 527. In Exercises 69 and 70, you are asked to show that the results obtained by these two procedures differ only by a constant.

EXAMPLE 6 Power of Tangent Is Even

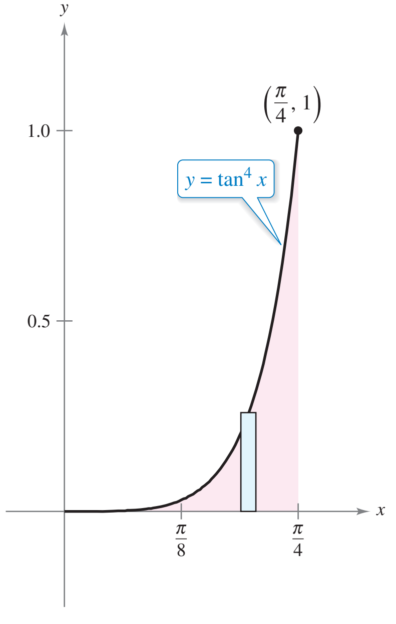
Evaluate $\int_0^{\pi/4} \tan^4 x \, dx$.

Solution Because there are no secant factors, you can begin by converting a tangent-squared factor to a secant-squared factor.

$$\begin{aligned}
\int \tan^4 x \, dx &= \int \tan^2 x(\tan^2 x) \, dx \\
&= \int \tan^2 x(\sec^2 x - 1) \, dx \\
&= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\
&= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \\
&= \frac{\tan^3 x}{3} - \tan x + x + C
\end{aligned}$$

Next, evaluate the definite integral.

$$\begin{aligned}
\int_0^{\pi/4} \tan^4 x \, dx &= \left[\frac{\tan^3 x}{3} - \tan x + x\right]_0^{\pi/4} \\
&= \frac{1}{3} - 1 + \frac{\pi}{4} \\
&\approx 0.119
\end{aligned}$$



The area of the region is approximately 0.119.
Figure 8.5

The area represented by the definite integral is shown in Figure 8.5. Try using Simpson's Rule to approximate this integral. With $n = 18$, you should obtain an approximation that is within 0.00001 of the actual value.

For integrals involving powers of cotangents and cosecants, you can follow a strategy similar to that used for powers of tangents and secants. Also, when integrating trigonometric functions, remember that it sometimes helps to convert the entire integrand to powers of sines and cosines.

EXAMPLE 7 **Converting to Sines and Cosines**

Find $\int \frac{\sec x}{\tan^2 x}\,dx$.

Solution Because the first four guidelines on page 527 do not apply, try converting the integrand to sines and cosines. In this case, you are able to integrate the resulting powers of sine and cosine as shown.

$$\begin{aligned}\int \frac{\sec x}{\tan^2 x}\,dx &= \int \left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{\sin x}\right)^2 dx\\ &= \int (\sin x)^{-2}(\cos x)\,dx\\ &= -(\sin x)^{-1} + C\\ &= -\csc x + C\end{aligned}$$

Integrals Involving Sine-Cosine Products with Different Angles

Integrals involving the products of sines and cosines of two *different* angles occur in many applications. In such instances, you can use the following product-to-sum identities.

$$\sin mx \sin nx = \frac{1}{2}(\cos[(m-n)x] - \cos[(m+n)x])$$

$$\sin mx \cos nx = \frac{1}{2}(\sin[(m-n)x] + \sin[(m+n)x])$$

$$\cos mx \cos nx = \frac{1}{2}(\cos[(m-n)x] + \cos[(m+n)x])$$

EXAMPLE 8 **Using Product-to-Sum Identities**

Find $\int \sin 5x \cos 4x\,dx$.

Solution Considering the second product-to-sum identity above, you can write

$$\begin{aligned}\int \sin 5x \cos 4x\,dx &= \frac{1}{2}\int (\sin x + \sin 9x)\,dx\\ &= \frac{1}{2}\left(-\cos x - \frac{\cos 9x}{9}\right) + C\\ &= -\frac{\cos x}{2} - \frac{\cos 9x}{18} + C.\end{aligned}$$

FOR FURTHER INFORMATION
To learn more about integrals involving sine-cosine products with different angles, see the article "Integrals of Products of Sine and Cosine with Different Arguments" by Sherrie J. Nicol in *The College Mathematics Journal.* To view this article, go to *MathArticles.com.*

8.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding an Indefinite Integral Involving Sine and Cosine In Exercises 1–12, find the indefinite integral.

1. $\int \cos^5 x \sin x\, dx$
2. $\int \cos^3 x \sin^4 x\, dx$
3. $\int \sin^7 2x \cos 2x\, dx$
4. $\int \sin^3 3x\, dx$
5. $\int \sin^3 x \cos^2 x\, dx$
6. $\int \cos^3 \frac{x}{3}\, dx$
7. $\int \sin^3 2\theta \sqrt{\cos 2\theta}\, d\theta$
8. $\int \frac{\cos^5 t}{\sqrt{\sin t}}\, dt$
9. $\int \cos^2 3x\, dx$
10. $\int \sin^4 6\theta\, d\theta$
11. $\int x \sin^2 x\, dx$
12. $\int x^2 \sin^2 x\, dx$

Using Wallis's Formulas In Exercises 13–18, use Wallis's Formulas to evaluate the integral.

13. $\int_0^{\pi/2} \cos^7 x\, dx$
14. $\int_0^{\pi/2} \cos^9 x\, dx$
15. $\int_0^{\pi/2} \cos^{10} x\, dx$
16. $\int_0^{\pi/2} \sin^5 x\, dx$
17. $\int_0^{\pi/2} \sin^6 x\, dx$
18. $\int_0^{\pi/2} \sin^8 x\, dx$

Finding an Indefinite Integral Involving Secant and Tangent In Exercises 19–32, find the indefinite integral.

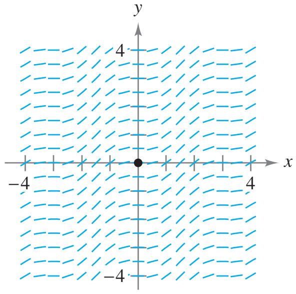
19. $\int \sec 4x\, dx$
20. $\int \sec^4 2x\, dx$
21. $\int \sec^3 \pi x\, dx$
22. $\int \tan^6 3x\, dx$
23. $\int \tan^5 \frac{x}{2}\, dx$
24. $\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2}\, dx$
25. $\int \tan^3 2t \sec^3 2t\, dt$
26. $\int \tan^5 2x \sec^4 2x\, dx$
27. $\int \sec^6 4x \tan 4x\, dx$
28. $\int \sec^2 \frac{x}{2} \tan \frac{x}{2}\, dx$
29. $\int \sec^5 x \tan^3 x\, dx$
30. $\int \tan^3 3x\, dx$
31. $\int \frac{\tan^2 x}{\sec x}\, dx$
32. $\int \frac{\tan^2 x}{\sec^5 x}\, dx$

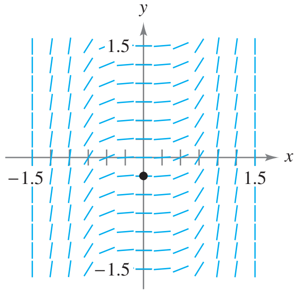
Differential Equation In Exercises 33–36, solve the differential equation.

33. $\frac{dr}{d\theta} = \sin^4 \pi\theta$
34. $\frac{ds}{d\alpha} = \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2}$
35. $y' = \tan^3 3x \sec 3x$
36. $y' = \sqrt{\tan x} \sec^4 x$

Slope Field In Exercises 37 and 38, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to *MathGraphs.com.*

37. $\frac{dy}{dx} = \sin^2 x,\ (0, 0)$
38. $\frac{dy}{dx} = \sec^2 x \tan^2 x,\ \left(0, -\frac{1}{4}\right)$





Slope Field In Exercises 39 and 40, use a computer algebra system to graph the slope field for the differential equation, and graph the solution through the specified initial condition.

39. $\frac{dy}{dx} = \frac{3 \sin x}{y},\ \ y(0) = 2$
40. $\frac{dy}{dx} = 3\sqrt{y} \tan^2 x,\ \ y(0) = 3$

Using Product-to-Sum Identities In Exercises 41–46, find the indefinite integral.

41. $\int \cos 2x \cos 6x\, dx$
42. $\int \cos 5\theta \cos 3\theta\, d\theta$
43. $\int \sin 2x \cos 4x\, dx$
44. $\int \sin(-7x) \cos 6x\, dx$
45. $\int \sin \theta \sin 3\theta\, d\theta$
46. $\int \sin 5x \sin 4x\, dx$

Finding an Indefinite Integral In Exercises 47–56, find the indefinite integral. Use a computer algebra system to confirm your result.

47. $\int \cot^3 2x\, dx$
48. $\int \tan^5 \frac{x}{4} \sec^4 \frac{x}{4}\, dx$
49. $\int \csc^4 3x\, dx$
50. $\int \cot^3 \frac{x}{2} \csc^4 \frac{x}{2}\, dx$
51. $\int \frac{\cot^2 t}{\csc t}\, dt$
52. $\int \frac{\cot^3 t}{\csc t}\, dt$
53. $\int \frac{1}{\sec x \tan x}\, dx$
54. $\int \frac{\sin^2 x - \cos^2 x}{\cos x}\, dx$
55. $\int (\tan^4 t - \sec^4 t)\, dt$
56. $\int \frac{1 - \sec t}{\cos t - 1}\, dt$

Evaluating a Definite Integral **In Exercises 57–64, evaluate the definite integral.**

57. $\int_{-\pi}^{\pi} \sin^2 x\, dx$

58. $\int_{0}^{\pi/3} \tan^2 x\, dx$

59. $\int_{0}^{\pi/4} 6 \tan^3 x\, dx$

60. $\int_{0}^{\pi/3} \sec^{3/2} x \tan x\, dx$

61. $\int_{0}^{\pi/2} \frac{\cos t}{1 + \sin t}\, dt$

62. $\int_{\pi/6}^{\pi/3} \sin 6x \cos 4x\, dx$

63. $\int_{-\pi/2}^{\pi/2} 3 \cos^3 x\, dx$

64. $\int_{-\pi/2}^{\pi/2} (\sin^2 x + 1)\, dx$

WRITING ABOUT CONCEPTS

65. Describing How to Find an Integral In your own words, describe how you would integrate $\int \sin^m x \cos^n x\, dx$ for each condition.

(a) m is positive and odd.

(b) n is positive and odd.

(c) m and n are both positive and even.

66. Describing How to Find an Integral In your own words, describe how you would integrate $\int \sec^m x \tan^n x\, dx$ for each condition.

(a) m is positive and even.

(b) n is positive and odd.

(c) n is positive and even, and there are no secant factors.

(d) m is positive and odd, and there are no tangent factors.

67. Comparing Methods Evaluate $\int \sin x \cos x\, dx$ using the given method. Explain how your answers differ for each method.

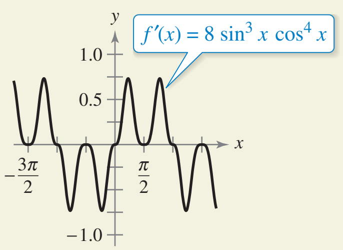
(a) Substitution where $u = \sin x$

(b) Substitution where $u = \cos x$

(c) Integration by parts

(d) Using the identity $\sin 2x = 2 \sin x \cos x$

68. HOW DO YOU SEE IT? Use the graph of f' shown in the figure to answer the following.



(a) Using the interval shown in the graph, approximate the value(s) of x where f is maximum. Explain.

(b) Using the interval shown in the graph, approximate the value(s) of x where f is minimum. Explain.

Comparing Methods **In Exercises 69 and 70, (a) find the indefinite integral in two different ways. (b) Use a graphing utility to graph the antiderivative (without the constant of integration) obtained by each method to show that the results differ only by a constant. (c) Verify analytically that the results differ only by a constant.**

69. $\int \sec^4 3x \tan^3 3x\, dx$

70. $\int \sec^2 x \tan x\, dx$

Area **In Exercises 71–74, find the area of the region bounded by the graphs of the equations.**

71. $y = \sin x, \quad y = \sin^3 x, \quad x = 0, \quad x = \frac{\pi}{2}$

72. $y = \sin^2 \pi x, \quad y = 0, \quad x = 0, \quad x = 1$

73. $y = \cos^2 x, \quad y = \sin^2 x, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$

74. $y = \cos^2 x, \quad y = \sin x \cos x, \quad x = -\frac{\pi}{2}, \quad x = \frac{\pi}{4}$

Volume **In Exercises 75 and 76, find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the x-axis.**

75. $y = \tan x, \quad y = 0, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$

76. $y = \cos \frac{x}{2}, \quad y = \sin \frac{x}{2}, \quad x = 0, \quad x = \frac{\pi}{2}$

Volume and Centroid **In Exercises 77 and 78, for the region bounded by the graphs of the equations, find (a) the volume of the solid formed by revolving the region about the x-axis and (b) the centroid of the region.**

77. $y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$

78. $y = \cos x, \quad y = 0, \quad x = 0, \quad x = \frac{\pi}{2}$

Verifying a Reduction Formula **In Exercises 79–82, use integration by parts to verify the reduction formula.**

79. $\int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx$

80. $\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\, dx$

81. $\int \cos^m x \sin^n x\, dx = -\frac{\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x\, dx$

82. $\int \sec^n x\, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx$

Using Formulas **In Exercises 83–86, use the results of Exercises 79–82 to find the integral.**

83. $\int \sin^5 x\, dx$

84. $\int \cos^4 x\, dx$

85. $\int \sec^4 (2\pi x/5)\,dx$ **86.** $\int \sin^4 x \cos^2 x\,dx$

87. Modeling Data The table shows the normal maximum (high) and minimum (low) temperatures (in degrees Fahrenheit) in Erie, Pennsylvania, for each month of the year. *(Source: NOAA)*

Month	Jan	Feb	Mar	Apr	May	Jun
Max	33.5	35.4	44.7	55.6	67.4	76.2
Min	20.3	20.9	28.2	37.9	48.7	58.5

Month	Jul	Aug	Sep	Oct	Nov	Dec
Max	80.4	79.0	72.0	61.0	49.3	38.6
Min	63.7	62.7	55.9	45.5	36.4	26.8

The maximum and minimum temperatures can be modeled by

$$f(t) = a_0 + a_1 \cos\frac{\pi t}{6} + b_1 \sin\frac{\pi t}{6}$$

where $t = 0$ corresponds to January 1 and a_0, a_1, and b_1 are as follows.

$$a_0 = \frac{1}{12}\int_0^{12} f(t)\,dt \qquad a_1 = \frac{1}{6}\int_0^{12} f(t)\cos\frac{\pi t}{6}\,dt$$

$$b_1 = \frac{1}{6}\int_0^{12} f(t)\sin\frac{\pi t}{6}\,dt$$

(a) Approximate the model $H(t)$ for the maximum temperatures. (*Hint:* Use Simpson's Rule to approximate the integrals and use the January data twice.)

(b) Repeat part (a) for a model $L(t)$ for the minimum temperature data.

(c) Use a graphing utility to graph each model. During what part of the year is the difference between the maximum and minimum temperatures greatest?

88. Wallis's Formulas Use the result of Exercise 80 to prove the following versions of Wallis's Formulas.

(a) If n is odd $(n \ge 3)$, then

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right).$$

(b) If n is even $(n \ge 2)$, then

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right).$$

89. Orthogonal Functions The **inner product** of two functions f and g on $[a, b]$ is given by

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx.$$

Two distinct functions f and g are said to be **orthogonal** if $\langle f, g\rangle = 0$. Show that the following set of functions is orthogonal on $[-\pi, \pi]$.

$$\{\sin x, \sin 2x, \sin 3x, \ldots, \cos x, \cos 2x, \cos 3x, \ldots\}$$

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90. Fourier Series The following sum is a *finite Fourier series.*

$$f(x) = \sum_{i=1}^{N} a_i \sin ix$$

$$= a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots + a_N \sin Nx$$

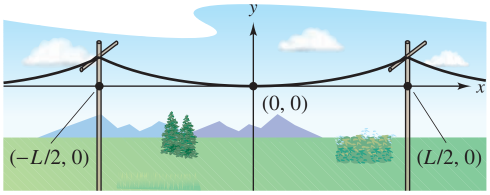
(a) Use Exercise 89 to show that the nth coefficient a_n is given by $a_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\sin nx\,dx$.

(b) Let $f(x) = x$. Find a_1, a_2, and a_3.

SECTION PROJECT

Power Lines

Power lines are constructed by stringing wire between supports and adjusting the tension on each span. The wire hangs between supports in the shape of a catenary, as shown in the figure.



Let T be the tension (in pounds) on a span of wire, let u be the density (in pounds per foot), let $g \approx 32.2$ be the acceleration due to gravity (in feet per second per second), and let L be the distance (in feet) between the supports. Then the equation of the catenary is $y = \frac{T}{ug}\left(\cosh\frac{ugx}{T} - 1\right)$, where x and y are measured in feet.

(a) Find the length of the wire between two spans.

(b) To measure the tension in a span, power line workers use the *return wave method.* The wire is struck at one support, creating a wave in the line, and the time t (in seconds) it takes for the wave to make a round trip is measured. The velocity v (in feet per second) is given by $v = \sqrt{T/u}$. How long does it take the wave to make a round trip between supports?

(c) The sag s (in inches) can be obtained by evaluating y when $x = L/2$ in the equation for the catenary (and multiplying by 12). In practice, however, power line workers use the "lineman's equation" given by $s \approx 12.075t^2$. Use the fact that

$$\cosh\frac{ugL}{2T} + 1 \approx 2$$

to derive this equation.

FOR FURTHER INFORMATION To learn more about the mathematics of power lines, see the article "Constructing Power Lines" by Thomas O'Neil in *The UMAP Journal.*